Math 246C Lecture 25 Notes

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1 Plurisubharmonic Functions and the $\overline{\partial}$ Problem in Several Complex Variables

1.1 Properties of plurisubharmonic functions

Let $\Omega \subseteq \mathbb{C}^n$ be open. Last time, we said that $u : \Omega \to [-\infty, \infty)$ is plurisubharmonic if

1. u is upper semicontinuous
2. for al $z \in \Omega$ and $w \in \mathbb{C}$, $\mathbb{C} \ni \tau \to u(z + \tau w)$ is subharmonic.

Example 1.1. Let $f \in \mathrm{Hol}(\Omega)$ for an open $\Omega \subseteq \mathbb{C}^n$. Then $\log |f|$ and $|f|^a$ are plirisubharmonic for $a > 0$.

Proposition 1.1. *Let $u \in C^2(\Omega)$ be real. Then u is plurisubharmonic if and only if for any $z \in \Omega$ and $w \in \mathbb{C}^n$,*

$$\sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0.$$

Proof. We have that u is plurisubharmonic iff $\Delta_\tau(u(z + \tau w)) \geq 0$:

$$\partial_\tau(u(z + \tau w)) = \sum_{j=1}^n \frac{\partial u}{\partial z_j}(z + \tau w) w_j.$$

$$\partial_{\overline{\tau}}(\partial_\tau(u(z + \tau w))) = \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z + \tau w) w_j \overline{w}_k \geq 0. \qquad \square$$

Remark 1.1. The Hermitian form $\mathcal{L}_u(w) = u''_{z,\overline{z}} \overline{w} \cdot w \geq 0$ is called the **Levi form** of u.

Plurisubharmonic functions have the following properties:

Proposition 1.2. *If $\Omega \subseteq \mathbb{C}^n$ is connected and $u \not\equiv -\infty$ is plurisubharmonic in Ω, then $u \in L^1_{\mathrm{loc}}$.*

Proof. Use the same argument as for subharmonic functions, using the sub-mean value property. If $n = 2$, let $D = D_1 \times D_2 \subseteq \mathbb{C}^2$ be a polydisc with $D_j = D(z_j^0, r_j)$. Then

$$\iint_D u(z_1, z_2)\, L(d(z_1, z_2)) \geq \int_{D_1} u(z_1, z_2^0)\, dm \geq m(D) u(z_1^0, z_2^0). \qquad \square$$

Proposition 1.3 (Regularization of plurisubharmonic functions)**.** *Let* $0 \leq \varphi \in C_0^\infty(\mathbb{C}^n)$ *be such that* $\int \varphi = 1$ *and* φ *depends only on* $|z_1|, \dots, |z_n|$. *Then* $u_\varepsilon = u * \varphi_\varepsilon \in C^\infty \cap \mathrm{PSH}$, *where* $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^{2n}} \varphi(z/\varepsilon)$, *and* $u_\varepsilon \downarrow u$ *as* $\varepsilon \downarrow 0$.

1.2 L^2-estimates for the $\overline{\partial}$-operator for several complex variables

Let $\Omega \subseteq \mathbb{C}^n$ be open. We will study $\overline{\partial} u = f$, where $u \in L^2_{\mathrm{loc}}$ and f is a 1-form: $f = \sum f_j\, d\overline{z}_j$.[1] Then

$$\overline{\partial} f = 0 \iff \frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\overline{\partial} z_j} \qquad \forall j, k, f_j \in L^2_{\mathrm{loc}}$$

in the weak sense.

We will develop a Hilbert space approach to this problem. Let $H_1 = L^2(\Omega, e^{-\varphi_1})$, where $\varphi_1 \in C^\infty(\Omega)$ is real. Let

$$H_2 = L^2_{(0,1)}(\Omega, e^{-\varphi_2}) = \{f = \sum_{j=1}^n f_j\, dz_j : f_j \in L^2(\Omega, e^{-\varphi_2})\}, \qquad \|f\|^2 = \sum \|f_j\|^2_{\varphi_2},$$

where $\varphi_2 \in C^\infty(\Omega)$. Consider the densely defined operator $T : H_1 \to H_2$ sending $u \mapsto \overline{\partial} u$, where

$$D(T) = \{u \in L^2(\Omega, e^{-\varphi_1}) \mid \overline{\partial} u \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \exists f_j \in L^2(\Omega, e^{\varphi_2}) \text{ s.t. } \frac{\partial u}{\partial \overline{z}_j} = f_j \text{ weakly}\}.$$

Definition 1.1. Let H_1, H_2 be Hilbert spaces. A linear map $T : H_1 \to H_2$ with domain $D(T)$ is **closed** if the graph of T, $G(T) = \{x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$ is closed.

In other words, if $x_n \in D(T)$ is such that $x_n \to x \in H_1$ and $Tx_n \to y$, then $x \in D(T)$, and $y = Tx$.

We have that $T = \overline{\partial} : L^2(\Omega, e^{-\varphi_1}) \to L^2_{(0,1)}(\Omega, e^{-\varphi_2})$ is closed. We have that $\mathrm{Ran}(T) \subseteq F = \{f \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \overline{\partial} f = 0 \text{ weakly}\} \subseteq H_2$, a closed subspace. We will try to show that $\mathrm{Ran}(T) = F$ for suitable φ_1, φ_2. Introduce the adjoint of T:

Definition 1.2. Let $T : H_1 \to H_2$ be linear and densely defined. We define the **adjoint** $T^* : H_2 \to H_1$ as follows:

$$D(T^*) = \{v \in H_2 : \exists f \in H_1 \text{ s.t. } \langle Tu, v \rangle_{H_2} = \langle u, f \rangle_{H_1} \ \forall u \in D(T)\}.$$

We let $T^* c = f$ ($D(T)$ is dense, so f is unique).

[1] This is sometimes called a $(0,1)$-form, as it has no z_j differentials.

Remark 1.2. Like T itself, the adjoint may be unbounded.

Proposition 1.4. *The adjoint satisfies the following property:*

1. *If T is closed and densely defined, then T^* is closed and densely defined.*
2. *$T^{**} = T$.*

Example 1.2. Let $H_1 = L^2(\Omega, e^{-\varphi_1})$, $H_2 = L^2_{(0,1)}(\Omega, e^{-\varphi_2})$. and $T = \overline{\partial}$. Then

$$D(\overline{\partial}^*) = \{v \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \forall u \in D(\overline{\partial}), \sum_{j=1}^n \int_\Omega \frac{\partial u}{\partial \overline{z}_j} \overline{v}_j e^{-\varphi_2}\, L(dz) = \int_\Omega u \overline{f} e^{-\varphi_1}\, L(dz)$$

$$\text{for some } f \in L^2(\Omega, e^{-\varphi_1})\}.$$

By integration by parts, $C^\infty_{0,(0,1)}(\Omega) \subseteq D(\overline{\partial}^*)$. If $v \in D(\overline{\partial}^*)$, we get taking $u \in C^\infty_0$ that $f = \overline{\partial}^* v = -\sum_{j=1}^n e^{\varphi_1} \partial_{z_j}(e^{-\varphi_2} v_j)$, where these are weak derivatives.

We have a closed $T : H_1 \to H_2$ where $\mathrm{Ran}(T) \subseteq F \subseteq H_2$ is closed. Next time, we will show the following.

Lemma 1.1. $\mathrm{Ran}(T) = F$ *if and only if there exists $C > 0$ such that $\|f\|_{H_2} \le C \|T^* f\|_{H_1}$ for all $f \in F \cap D(T^*)$.*